An inverse scattering problem for the Klein-Gordon equation with a classical source in quantum field theory

Hironobu Sasaki *
and
Akito Suzuki †

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Abstract

An inverse scattering problem for a quantized scalar field ϕ obeying a linear Klein-Gordon equation

$$(\Box + m^2 + V)\phi = J \quad \text{in } \mathbb{R} \times \mathbb{R}^3$$

is considered, where V is a repulsive external potential and J an external source J. We prove that the scattering operator $\mathscr{S} = \mathscr{S}(V, J)$ associated with ϕ uniquely determines V. Assuming that J is of the form $J(t, x) = j(t)\rho(x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, we represent ρ (resp. j) in terms of j (resp. ρ) and $\mathscr{S}$.

1 Introduction

We consider an inverse scattering problem for a quantized scalar field ϕ interacting with an external potential V and an external source J (see, e.g., [5, 11]) which obeys the Klein-Gordon equation

$$(\Box + m^2 + V(x))\phi(t, x) = J(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^3. \tag{1.1}$$

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*Department of Mathematics and Informatics, Chiba 263-8522, Japan, E-mail: sasaki@math.s.chiba-u.ac.jp

†Department of Engineering, Shinshu University, Nagano, 380-8553, Japan E-mail: akito@shinshu-u.ac.jp

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Here $\square = \frac{\partial^2}{\partial t^2} - \Delta$, Δ is the Laplacian in $\mathbb{R}^3$, $m > 0$ and J, V are real functions. $\boldsymbol{\phi}(t,x)$ and its conjugate field $\boldsymbol{\pi}(t,x) = \frac{\partial}{\partial t}\boldsymbol{\phi}(t,x)$ are operator valued distributions (see, e.g., [18]). A typical example of (1.1) is the nucleon-pion interaction, that is, $\boldsymbol{\phi}$ describes the pion field and J the distribution function of the nucleons (see, e.g., [5]).

Under suitable conditions, one can show that the asymptotic fields

$$\boldsymbol{\phi}_{\mathrm{out/in}}(t,x) = \text{s-}\lim_{s \to \pm\infty} \boldsymbol{\phi}_s(t,x), \qquad (1.2)$$

$$\boldsymbol{\pi}_{\mathrm{out/in}}(t,x) = \text{s-}\lim_{s \to \pm\infty} \boldsymbol{\pi}_s(t,x) \qquad (1.3)$$

exist. Here $\boldsymbol{\pi}_s(t,x) = \frac{\partial}{\partial t}\boldsymbol{\phi}_s(t,x)$ and $\boldsymbol{\phi}_s(t,x)$ is the solution of the free Klein-Gordon equation with the initial condition: $\boldsymbol{\phi}_s(s,x) = \boldsymbol{\phi}(s,x)$ and $\boldsymbol{\pi}_s(s,x) = \boldsymbol{\pi}(s,x)$. Suppose that $\boldsymbol{\phi}_{\mathrm{in}}(x) = \boldsymbol{\phi}_{\mathrm{in}}(0,x)$ and $\boldsymbol{\pi}_{\mathrm{in}}(x) = \boldsymbol{\pi}_{\mathrm{in}}(0,x)$ give the Fock representation of the canonical commutation relations (CCR):

$$[\boldsymbol{\phi}_{\mathrm{in}}(x), \boldsymbol{\pi}_{\mathrm{in}}(y)] = i\delta(x-y), \qquad (1.4)$$

$$[\boldsymbol{\phi}_{\mathrm{in}}(x), \boldsymbol{\phi}_{\mathrm{in}}(y)] = [\boldsymbol{\pi}_{\mathrm{in}}(x), \boldsymbol{\pi}_{\mathrm{in}}(y)] = 0. \qquad (1.5)$$

See Section 3 for the detail. The scattering operator $\mathscr{S} = \mathscr{S}(V, J)$ is defined by the following relations (up to a constant factor):

$$\mathscr{S}^{-1}\boldsymbol{\phi}_{\mathrm{in}}(x)\mathscr{S} = \boldsymbol{\phi}_{\mathrm{out}}(x), \quad \mathscr{S}^{-1}\boldsymbol{\pi}_{\mathrm{in}}(x)\mathscr{S} = \boldsymbol{\pi}_{\mathrm{out}}(x). \qquad (1.6)$$

We prove that $\mathscr{S}$ uniquely determines V. Suppose that $J(t,x) = j(t)\rho(x)$. Then we show that ρ (resp. j) is uniquely determined by $\mathscr{S}$ and j (resp. ρ).

To state our results precisely, we introduce several assumptions. We set $\mathfrak{h} = L^2(\mathbb{R}^3; dx)$ and assume the following:

Assumption 1.1. *The potential function* $V : \mathbb{R}^3 \to \mathbb{R}$ *is non negative and satisfies* $V \in H^2(\mathbb{R}^3)$.

Then the multiplication operator V acting in $\mathfrak{h}$ is infinitesimally small with respect to $h_0 = -\Delta$ since $V \in L^2(\mathbb{R}^3)$. Hence the operator

$$h = h_0 + V$$

is self-adjoint with the domain $D(h) = D(h_0) = H^2(\mathbb{R}^3)$. Since V is relative compact with respect to h_0, i.e., $V(h_0 + 1)^{-1}$ is compact, and V is positive, the spectrum of h is $\sigma(h) = \sigma_{\mathrm{ess}}(h) = [0, \infty)$. The condition $V \in H^2(\mathbb{R}^3)$ allows us to construct the solution of (1.1) by a Bogoliubov transformation (see Lemma 2.2 and Proposition 2.4).

We set

$$\omega = \varphi(h), \quad \omega_0 = \varphi(h_0),$$

where $\varphi(s) = \sqrt{s + m^2}$.

Assumption 1.2. *We assume that h has no positive eigenvalue and*

$$\int_0^\infty dR \|V(x)(-\Delta + 1)^{-1} F(|x| \ge R)\| < \infty, \qquad (1.7)$$

where

$$F(|x| \ge R) = \begin{cases} 1 & \text{if } |x| \ge R, \\ 0 & \text{if } |x| < R. \end{cases}$$

We make some comments on Assumption 1.2:

- By (1.7), the following limits exist

$$w_\pm := \text{s-}\lim_{t \to \pm\infty} e^{ith} e^{-ith_0}$$

  and the intertwining property $hw_\pm = w_\pm h_0$ holds. By Enss and Weder [3], we see that the scattering map for the Schrödinger operator defined by

$$\mathcal{V}_{\text{SR}} \ni V \mapsto S(V) = w_+^* w_-$$

  is injective, where

$$\mathcal{V}_{\text{SR}} = \{V : \mathbb{R}^3 \to \mathbb{R} \mid V \text{ is Kato-small in } \mathfrak{h} \text{ and satisfies } (1.7)\}.$$

- Since h has no positive eigenvalue, (1.7) implies that h has purely absolutely continuous spectrum. In particular, we have $w_\pm^* w_\pm = w_\pm w_\pm^* = I$ on $\mathfrak{h}$.

- We see that the following limits exist:

$$\text{s-}\lim_{t \to \pm\infty} e^{it\omega} e^{-it\omega_0}. \qquad (1.8)$$

  For a proof of the existence, see Appendix A.2. By [19, Theorem 1], above limits (1.8) are equal to $w_\pm$, respectively, i.e., the invariance principal holds for φ.

Assumption 1.3. *The function $J : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ satisfies*

(a) *For each $t \in \mathbb{R}$, the function $J_t(x) := J(t, x)$ satisfies $J_t \in H^{-1/2}(\mathbb{R}^3)$.*

(b) *The vector valued function $\mathbb{R} \ni t \mapsto e^{-it\omega} \omega^{-1/2} J_t \in \mathfrak{h}$ satisfies*

$$\int_{-\infty}^\infty dt \|\omega^{-1/2} J_t\|_{\mathfrak{h}} < \infty.$$

We say that $V \in \mathcal{V}$ if V satisfies Assumptions 1.1 and 1.2 and that $J \in \mathcal{J}$ if J satisfies Assumption 1.3.

Theorem 1.1. *Suppose that $V, V' \in \mathcal{V}$ and $J, J' \in \mathcal{J}$. If $\mathscr{S}(V, J) = \mathscr{S}(V', J')$, then:*

(i) $S(V) = S(V')$,

(ii) $V = V'$,

(iii) $\int_{-\infty}^{+\infty} ds e^{-is\omega} J_t = \int_{-\infty}^{+\infty} ds e^{-is\omega} J'_t$.

By the above theorem, we immediately see the following:

Corollary 1.2. *Let $J \in \mathcal{J}$ be given. Then the map $\mathcal{V} \ni V \mapsto \mathscr{S}(V, J)$ is injective.*

Proof of Theorem 1.1. (i) will be proved in Theorem 4.1. Since $\mathcal{V} \subset \mathcal{V}_{\mathrm{SR}}$, (ii) follows from the injectivity of the map $V \mapsto S(V)$. We will prove (iii) in Proposition 4.5. □

In order to recover the external source J, we henceforth suppose that $J \in \mathcal{J}$ is expressed by

$$J(t, x) = j(t) \times \rho(x), \tag{1.9}$$

where $j \in L^1(\mathbb{R})$ and $\rho \in H^{-1/2}(\mathbb{R}^3)$. Let

$$F(t, f) = (\Omega_{\mathrm{in}}, \phi_{\mathrm{out}}(t, f)\Omega_{\mathrm{in}}), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where Ω_{in} is the Fock vacuum. From (1.6), we see that $F(t, f)$ is uniquely determined by $\mathscr{S}$. One can show the following:

(1) For a given function j such that the Fourier transform $\hat{j}$ of j is real analytic, we represent ρ in terms of j and $F(t, f)$. See Theorem 4.6 for the detail.

(2) Let ρ be a given function and assume that j satisfies the following:

$$\text{For some } \delta > 0, \quad e^{\delta|t|} j(t) \in L^1(\mathbb{R}_t). \tag{1.10}$$

Then we express j by means of ρ and $F(t, f)$. See Theorem 4.8 for the detail.

From the reconstruction formulas above, we observe that the scattering operator $\mathscr{S}(V, j \times \rho)$ uniquely determines j and ρ:

Theorem 1.3. *Let $V \in \mathcal{V}$ and $j \times \rho$, $j' \times \rho' \in \mathcal{J}$. Suppose that $j, j' \in L^1(\mathbb{R})$ and $\rho, \rho' \in H^{-1/2}(\mathbb{R})$. Then:*

(i) Assume that that $\hat{j}$ is real analytic. Then $\rho = \rho'$ if $\mathscr{S}(V, j \times \rho) = \mathscr{S}(V, j \times \rho')$.

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(ii) Assume that (1.10) holds. Then $j = j'$ if $\mathscr{S}(V, j \times \rho) = \mathscr{S}(V, j' \times \rho)$.

Proof. See Theorem 4.6. ρ is uniquely determined by j and a function z defined in (4.6). z is uniquely determined by $F(t, f)$. Since, as was noted above, $\mathscr{S}$ determines $F(t, f)$ uniquely, we see that $\rho \mapsto \mathscr{S}(V, j \times \rho)$ is injective. Hence (i) holds. (ii) is proved similarly. $\square$

This paper is organized as follows. Section 2 is devoted to some mathematical preliminaries. In Subsections 2.1 and 2.2, we review well-known facts. The quantized Klein-Gordon field is constructed in Subsection 2.3 and the wave operator in Subsection 2.4.. In Section 3, we discuss the scattering theory and define the scattering operator $\mathscr{S}$. Section 4 deals with the inverse scattering problem. In Subsection 4.1, we show the uniqueness of V. The reconstruction formulas of ρ and j are given in Subsections 4.2 and 4.3, respectively. In Appendix, we prove Lemmas 2.2 and 2.6.

2 Preliminary

In general we denote the inner product and the associated norm of a Hilbert space $\mathcal{L}$ by $(*, \cdot)_{\mathcal{L}}$ and $\| \cdot \|_{\mathcal{L}}$, respectively. The inner product is linear in $\cdot$ and antilinear in $*$. If there is no danger of confusion, we omit the subscript $\mathcal{L}$ in $(\cdot, \cdot)_{\mathcal{L}}$ and $\| \cdot \|_{\mathcal{L}}$. For a linear operator T on $\mathcal{L}$, we denote the domain of T by $D(T)$ and, if $D(T)$ is dense in $\mathcal{H}$, the adjoint of T by T^*.

2.1 Boson Fock space

We first recall the abstract Boson Fock space and operators therein. The Boson Fock space over a Hilbert space $\mathfrak{h}$ is defined by

$$
\Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \bigotimes_{\mathrm{s}}^{n} \mathfrak{h}
$$
$$
= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\middle|\, \Psi^{(n)} \in \bigotimes_{\mathrm{s}}^{n} \mathfrak{h}, \quad \sum_{n=0}^{\infty} \left\| \Psi^{(n)} \right\|_{\otimes_{\mathrm{s}}^{n}\mathfrak{h}}^{2} < \infty \right\},
$$

where $\otimes_{\mathrm{s}}^{n}\mathfrak{h}$ denotes the symmetric tensor product of $\mathfrak{h}$ with the convention $\otimes_{\mathrm{s}}^{0}\mathfrak{h} = \mathbb{C}$.

The creation operator $c^*(f)$ ($f \in \mathfrak{h}$) acting in $\Gamma(\mathfrak{h})$ is defined by

$$
(c^*(f)\Psi)^{(n)} = \sqrt{n} S_n \left(f \otimes \Psi^{(n-1)} \right)
$$

with the domain

$$D(c^*(f)) = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \,\middle|\, \sum_{n=0}^\infty n \left\| S_n \left(f \otimes \Psi^{(n-1)} \right) \right\|_{\otimes_s^n \mathfrak{h}}^2 < \infty \right\},$$

where S_n denotes the symmetrization operator on $\otimes^n \mathfrak{h}$ satisfying $S_n = S_n^* = S_n^2$ and $S_n(\otimes^n \mathfrak{h}) = \otimes_s^n \mathfrak{h}$.

The annihilation operator $c(f)$ ($f \in \mathfrak{h}$) is defined by the adjoint of $c^*(f)$, i.e., $c(f) := c^*(f)^*$. By definition, $c^*(f)$ (resp. $c(f)$) is linear (resp. antilinear) in $f \in \mathfrak{h}$. As is well known, the creation and annihilation operators leave the finite particle subspace

$$D_{\mathrm{f}} = \bigcup_{m=1}^\infty \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \mid \Psi^{(n)} = 0, \ \ n \geq m \right\}$$

invariant. The canonical commutation relations (CCR) hold on D_{f}:

$$[c(f), c^*(g)] = (f, g)_{\mathfrak{h}}, \quad [c(f), c(g)] = [c^*(f), c^*(g)] = 0. \tag{2.1}$$

It follows from (2.1) that

$$\|c^*(f)\Psi\|^2 = \|f\|^2 \|\Psi\|^2 + \|c(f)\Psi\|^2, \quad \Psi \in D_{\mathrm{f}}. \tag{2.2}$$

The Segal field operator $\tau(f) = \frac{1}{\sqrt{2}}(c(f) + c^*(f))$ ($f \in \mathfrak{h}$) is essentially self-adjoint on D_{f}. We denote its closure by the same symbol. By (2.1), the following equation holds

$$\|c^*(f)\Psi\|^2 = \frac{1}{2}(\|\tau(f)\Psi\|^2 + \|\tau(if)\Psi\|^2 + \|f\|^2 \|\Psi\|^2), \quad \Psi \in D_{\mathrm{f}}. \tag{2.3}$$

Since D_{f} is a core for $c(f)$, $c^*(f)$ and $\tau(f)$ ($f \in \mathfrak{h}$), we observe from (2.2) and (2.3) that

$$D(\tau(f)) \cap D(\tau(if)) = D(c(f)) = D(c^*(f)). \tag{2.4}$$

Hence the following operator equalities hold true:

$$c(f) = \frac{1}{\sqrt{2}}(\tau(f) + i\tau(if)),$$

$$c^*(f) = \frac{1}{\sqrt{2}}(\tau(f) - i\tau(if)).$$

Let

$$\tilde{D} := \bigcap_{f \in \mathfrak{h}} D(c(f)).$$

Since $\tilde{D} \supset D_{\mathrm{f}}$, $\tilde{D}$ is dense in $\Gamma(\mathfrak{h})$. From (2.4), we observe that

$$\tilde{D} = \bigcap_{f \in \mathfrak{h}} D(c^*(f)) = \bigcap_{f \in \mathfrak{h}} D(\tau(f)). \tag{2.5}$$

The Fock vacuum $\Omega = \{\Omega^{(n)}\}_{n=0}^{\infty} \in \Gamma(\mathfrak{h})$ is defined by $\Omega^{(0)} = 1$ and $\Omega^{(n)} = 0$ $(n \geq 1)$, which satisfies

$$c(f)\Omega = 0, \quad f \in \mathfrak{h}. \tag{2.6}$$

Ω is a unique vector satisfying (2.6) up to a constant factor.

Let A be a contraction operator on $\mathfrak{h}$, i.e., $\|A\| \leq 1$. We define a contraction operator $\Gamma(A)$ on $\Gamma(\mathfrak{h})$ by

$$(\Gamma(A)\Psi)^{(n)} = (\otimes^n A)\,\Psi^{(n)}, \quad \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty}$$

with the convention $\otimes^0 A = 1$. If U is unitary, i.e. $U^{-1} = U^*$, then $\Gamma(U)$ is also unitary and satisfies $\Gamma(U)^* = \Gamma(U^*)$ and

$$\Gamma(U)c(f)\Gamma(U)^* = c(Uf), \quad \Gamma(U)c^*(f)\Gamma(U)^* = c^*(Uf).$$

For a self-adjoint operator T on $\mathfrak{h}$, i.e., $T = T^*$, $\{\Gamma(e^{itT})\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\Gamma(\mathfrak{h})$. Then, by the Stone theorem, there exists a unique self-adjoint operator $d\Gamma(T)$ such that

$$\Gamma(e^{itT}) = e^{itd\Gamma(T)}.$$

The number operator N_{f} is defined by $d\Gamma(1)$.

2.2 Bogoliubov transformations

Let $\mathcal{H}$ be the direct sum of two Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, where $\mathcal{H}_+ = \mathfrak{h}$ and $\mathcal{H}_-$ is a copy of it:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \left\{ v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \,\middle|\, v_+, v_- \in \mathfrak{h} \right\}.$$

We denote by P_+ (resp. P_-) the projection from $\mathcal{H}$ onto $\mathcal{H}_+$ (resp. $\mathcal{H}_-$):

$$P_+ \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} v_+ \\ 0 \end{bmatrix}, \quad P_- \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} 0 \\ v_- \end{bmatrix}.$$

A vector $v \in \mathrm{Ran}P_+ = \mathcal{H}_+$ is identified with one in $\mathfrak{h}$: $\begin{bmatrix} v_+ \\ 0 \end{bmatrix} = v_+ \in \mathfrak{h}$. We define an involution Q on $\mathcal{H}$ by

$$Q = P_+ - P_-$$

and a conjugation C on $\mathcal{H}$ by

$$C \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} \bar{v}_- \\ \bar{v}_+ \end{bmatrix},$$

where $\bar{f}$ stands for the complex conjugation of f in $\mathfrak{h}$, i.e., $\bar{f}(x) = \overline{f(x)}$, a.e. $x \in \mathbb{R}^3$.

A bounded operator A on $\mathcal{H}$ is written as

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix},$$

where $A_{\epsilon\epsilon'} = P_\epsilon A P_{\epsilon'}$ ($\epsilon, \epsilon' = +$ or $-$). Then we observe that

$$A \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} A_{++}v_+ + A_{+-}v_- \\ A_{-+}v_+ + A_{--}v_- \end{bmatrix}$$

and $A_{\epsilon\epsilon'}^* = (A_{\epsilon\epsilon'})^*$. We introduce field operators defined on $\tilde{D}$ by

$$\psi(v) = c(P_+ v) + c^*(CP_- v)$$
$$= c(v_+) + c^*(\bar{v}_-), \quad v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \in \mathcal{H}.$$

One observes that $\psi(v)^* = \psi(Cv)$ on $\tilde{D}$ and hence that $\psi(v)$ is closable. We denote its closure by the same symbol. Let $\mathcal{H}_C$ be the set of vectors satisfying $Cv = v$:

$$\mathcal{H}_C = \left\{ v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \,\middle|\, v_+ = \bar{v}_- \in \mathfrak{h} \right\}.$$

Clearly, for any $v = \begin{bmatrix} f \\ \bar{f} \end{bmatrix} \in \mathcal{H}_C$ ($f \in \mathfrak{h}$), the operator $\psi(v)$ is essentially self-adjoint on D_{f} and is equal to $\sqrt{2}\tau(f)$. By (2.5), we see that

$$\tilde{D} = \bigcap_{v \in \mathcal{H}_C} D(\psi(v)).$$

Note that, for any $v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \in \mathcal{H}$, the vectors

$$v + Cv = \begin{bmatrix} v_+ + \bar{v}_- \\ v_- + \bar{v}_+ \end{bmatrix}, \quad i(v - Cv) = \begin{bmatrix} iv_+ - i\bar{v}_- \\ iv_- - i\bar{v}_+ \end{bmatrix}$$

belong to $\mathcal{H}_C$ and the following holds on $\tilde{D}$:

$$\psi(v) = \frac{1}{2}\psi(v + Cv) + \frac{i}{2}\psi(i(v - Cv)).$$

It is straightforward from (2.1) that

$$[\psi(u), \psi(v)^*] = (u, Qv)$$

holds on D_{f}. The following is well known (see, e.g., [13]):

Lemma 2.1. *(1) Let U be a bounded operator on $\mathcal{H}$ satisfying*

$$CU = UC, \quad UQU^* = U^*QU = Q \qquad (2.7)$$

Then there exists a unitary operator $\mathscr{U}$ such that

$$e^{i\psi(U^*v)} = \mathscr{U}^* e^{i\psi(v)} \mathscr{U}, \quad \mathscr{U} e^{i\psi(v)} \mathscr{U}^* = e^{i\psi(QUQv)}, \quad v \in \mathcal{H}_C$$

if and only if U_{-+} is Hilbert-Schmidt. In this case, $\mathscr{U}$ leaves $\tilde{D}$ invariant.

(2) Let l be a linear functional from $\mathcal{H}$ to $\mathbb{C}$. Then there exists a unitary operator $\mathscr{U}_l$ such that

$$e^{i(\psi(u)+l(u))} = \mathscr{U}_l^* e^{i\psi(u)} \mathscr{U}_l, \quad v \in \mathcal{H}_C$$

if and only if there exists a $u_l \in \mathcal{H}_C$ such that

$$l(u) = i(v_l, Qu), \quad u \in \mathcal{H}_C.$$

In this case, $\mathscr{U}_l$ leaves $\tilde{D}$ invariant and $\mathscr{U}_l = e^{-i\psi(v_l)}$.

2.3 Quantized Klein Gordon equation

Let $\mathfrak{h} = L^2(\mathbb{R}^3; dx)$ and $h_0 = -\Delta$ with the domain $D(h_0) = D(-\Delta) = H^2(\mathbb{R}^3)$. Then we define the Schrödinger operator h by

$$h = -\Delta + V(x) \qquad (2.8)$$

with the potential $V : \mathbb{R}^3 \mapsto \mathbb{R}$ satisfying Assumption 1.1. h is self-adjoint with the domain $D(h) = D(h_0)$. Let

$$\omega_0 := (h_0 + m^2)^{1/2} \quad \text{and} \quad \omega := (h + m^2)^{1/2}.$$

The free field Hamiltonian H_{f} is defined by

$$H_{\mathrm{f}} = d\Gamma(\omega_0).$$

Since ω_0 and ω are strictly positive, ω_0^{-1} and ω^{-1} is bounded. Note that $\omega_0^\theta \omega^{-\theta}$ and $\omega^\theta \omega_0^{-\theta}$ are bounded operators for any $0 \leq \theta \leq 1$. Indeed, since $h_0 \leq h$, we observe that $\|\omega_0^\theta f\| \leq \|\omega^\theta f\|$. Hence $\omega_0^\theta \omega^{-\theta}$ is bounded. On the other hand, since $D(h) = D(h_0)$, it follows from the closed graph theorem that $\|hf\| \leq C\|(h_0+1)f\|$ with some $C > 0$. Hence we observe that $\omega^\theta \omega_0^{-\theta}$ is bounded. From this fact, we see that $\omega^{-\theta} \omega_0^\theta$ and $\omega_0^{-\theta} \omega^\theta$ can be extended to bounded operators on $\mathfrak{h}$. We denote the extended operators by the same symbols. The following holds.

Lemma 2.2. *Suppose that Assumption 1.1 holds. Then the operators $\omega_0^{1/2}\omega^{-1/2} - 1$, $\omega_0^{-1/2}\omega^{1/2} - 1$ and $\omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2}$ are Hilbert-Schmidt.*

Proof. See Appendix A.1. □

For real $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$, we set

$$\phi_0(f) = \psi(u_0) \quad \text{and} \quad \pi_0(g) = \psi(v_0),$$

where

$$u_0 = \begin{bmatrix} \omega_0^{-1/2} f/\sqrt{2} \\ \omega_0^{-1/2} f/\sqrt{2} \end{bmatrix} \quad \text{and} \quad v_0 = \begin{bmatrix} i\omega_0^{+1/2} g/\sqrt{2} \\ -i\omega_0^{+1/2} g/\sqrt{2} \end{bmatrix}.$$

For non real $f \in H^{-1/2}(\mathbb{R}^3)$ (resp. $g \in H^{1/2}(\mathbb{R}^3)$), $\phi_0(f)$ (resp. $\pi_0(g)$) is defined by of $\phi_0((\text{Re}f) + i\phi_0(\text{Im}f)$ (resp. $\pi_0((\text{Re}g) + i\pi_0(\text{Im}g)$). Note that for non real $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$, $\phi_0(f)$ and $\pi_0(g)$ are non self-adjoint and the following equations hold on $\tilde{D}$:

$$\phi_0(f) = \frac{1}{\sqrt{2}}\left(c^*(\omega_0^{-1/2}f) + c(\omega_0^{-1/2}\bar{f})\right),$$

$$\pi_0(g) = \frac{i}{\sqrt{2}}\left(c^*(\omega_0^{+1/2}g) - c(\omega_0^{+1/2}\bar{g})\right).$$

By (2.1), one can show that the CCR holds on D_{f}:

$$[\phi_0(f), \pi_0(g)] = i(\bar{f}, g), \quad [\phi_0(f), \phi_0(\tilde{f})] = [\pi_0(g), \pi_0(\tilde{g})] = 0, \quad (2.9)$$

for any $f, \tilde{f} \in H^{-1/2}(\mathbb{R}^3)$ and $g, \tilde{g} \in H^{1/2}(\mathbb{R}^3)$. It holds from (2.9) that

$$\|\phi_0(f)\Psi\|^2 = \|\phi_0(\text{Re}f)\Psi\|^2 + \|\phi_0(\text{Re}f)\|^2,$$

$$\|\pi_0(g)\Psi\|^2 = \|\pi_0(\text{Re}g)\Psi\|^2 + \|\pi_0(\text{Re}g)\|^2, \quad \Psi \in D_{\text{f}}.$$

Hence we observe that, for non real $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$, $\phi_0(f)$ and $\pi_0(g)$ are closed on the natural domain.

We introduce the bounded operator on $\mathcal{H}$ by

$$U(t) = \begin{bmatrix} U_{++}(t) & U_{+-}(t) \\ U_{-+}(t) & U_{--}(t) \end{bmatrix}, \quad t \in \mathbb{R},$$

where

$$U_{++}(t) = \frac{1}{2}\left(\omega_0^{-1/2}\cos(t\omega)\omega_0^{1/2} + \omega_0^{1/2}\cos(t\omega)\omega_0^{-1/2}\right)$$

$$- \frac{i}{2}\left(\omega_0^{1/2}\omega^{-1/2}\sin(t\omega)\omega^{-1/2}\omega_0^{1/2}\right.$$

$$+ \omega_0^{-1/2}\omega^{1/2}\sin(t\omega)\omega^{1/2}\omega_0^{-1/2}\right)$$

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and

$$U_{-+}(t) = \frac{1}{2}\left(\omega_0^{1/2}\cos(t\omega)\omega_0^{-1/2} - \omega_0^{-1/2}\cos(t\omega)\omega_0^{1/2}\right)$$
$$- \frac{i}{2}\left(\omega_0^{1/2}\omega^{-1/2}\sin(t\omega)\omega^{-1/2}\omega_0^{1/2}\right.$$
$$\left. - \omega_0^{-1/2}\omega^{1/2}\sin(t\omega)\omega^{1/2}\omega_0^{-1/2}\right)$$

with $U_{--}(t) = \overline{U_{++}(t)}$ and $U_{+-}(t) = \overline{U_{-+}(t)}$. Here, for a linear operator A, we define $\bar{A}$ by $\bar{A}f = \overline{A\bar{f}}$.

Lemma 2.3. *Suppose that Assumption 1.1 holds. Then there exists a family of unitary operators $\mathscr{U}_t$ on $\Gamma(\mathfrak{h})$ such that $\mathscr{U}_t$ maps $\tilde{D}$ to $\tilde{D}$ and for $v \in \mathcal{H}_C$*

$$e^{i\psi(U(t)^*v)} = \mathscr{U}_t^* e^{i\psi(v)}\mathscr{U}_t, \quad \mathscr{U}_t e^{i\psi(v)}\mathscr{U}_t^* = e^{i\psi(QU(t)Qv)}.$$

Proof. By direct calculation, we observe that $U(t)$ satisfies (2.7) with $U = U(t)$. We note that $2U_{-+}(t)$ is equal to

$$(\omega_0^{1/2}\omega^{-1/2} - 1)\cos(t\omega)\omega^{1/2}\omega_0^{-1/2}$$
$$+ \cos(t\omega) \cdot \omega^{-1/2}\omega_0^{1/2} \cdot \omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2}$$
$$+ (1 - \omega_0^{-1/2}\omega^{1/2})\cos(t\omega)\omega^{-1/2}\omega_0^{1/2}$$
$$- i(\omega_0^{1/2}\omega^{-1/2} - 1)\sin(t\omega) \cdot \omega^{-1/2}\omega_0^{1/2}$$
$$- i\sin(t\omega) \cdot \omega^{-1/2}\omega_0^{1/2} \cdot \omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2}$$
$$- i(1 - \omega_0^{-1/2}\omega^{1/2})\sin(t\omega) \cdot \omega^{1/2}\omega_0^{-1/2}.$$

By Lemma 2.2, $\omega_0^{1/2}\omega^{-1/2} - 1$, $\omega_0^{-1/2}\omega^{1/2} - 1$ and $\omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2}$ are Hilbert-Schmidt, so is $U_{-+}(t)$. $\qquad\square$

Let

$$g_t = -\frac{1}{\sqrt{2}}\int_0^t ds\,\omega_0^{-1/2}\cos[(t-s)\omega]J_s$$
$$+ \frac{i}{\sqrt{2}}\int_0^t ds\,\omega_0^{1/2}\frac{\sin[(t-s)\omega]}{\omega}J_s$$

and

$$j_t = \begin{bmatrix} g_t \\ \bar{g}_t \end{bmatrix}.$$

For $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$, we define field operators

$$\phi(t,f) = \mathscr{U}(t)^*\phi_0(f)\mathscr{U}(t) \quad \text{and} \quad \pi(t,g) = \mathscr{U}(t)^*\pi_0(g)\mathscr{U}(t),$$

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where
$$\mathscr{U}(t) = e^{-i\psi(j_t)}\mathscr{U}_t.$$

The following propositions are standard:

Proposition 2.4. *Suppose that Assumptions 1.1 and 1.3 hold. Then it holds that, $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$,*

$$\phi(t, f) = \psi(u(t)) + \left(\int_0^t ds \frac{\sin[(t-s)\omega]}{\omega} J_s, f\right),$$

$$\pi(t, f) = \psi(v(t)) - \left(\int_0^t ds \cos[(t-s)\omega] J_s, f\right),$$

where

$$u(t) = \begin{bmatrix} \omega_0^{-1/2}(\cos(t\omega) + i\omega_0 \sin(t\omega)\omega^{-1})\bar{f}/\sqrt{2} \\ \omega_0^{-1/2}(\cos(t\omega) - i\omega_0 \sin(t\omega)\omega^{-1})\bar{f}/\sqrt{2} \end{bmatrix},$$

$$v(t) = \begin{bmatrix} i\omega_0^{1/2}(\cos(t\omega) + i\omega_0^{-1} \sin(t\omega)\omega)\bar{f}/\sqrt{2} \\ -i\omega_0^{1/2}(\cos(t\omega) - i\omega_0^{-1} \sin(t\omega)\omega)\bar{f}/\sqrt{2} \end{bmatrix}.$$

Proposition 2.5. *Suppose that Assumptions 1.1 and 1.3 hold. Let $\Psi \in D(N_{\mathrm{f}}^{1/2})$ and $f \in H^{-1/2}(\mathbb{R}^3) \cap H^{3/2}(\mathbb{R}^3)$. Then $\phi(0, f) = \phi_0(f)$, $\pi(0, f) = \pi_0(f)$ and*

$$\frac{d}{dt}\phi(t, f)\Psi = \pi(t, f)\Psi,$$

$$\frac{d^2}{dt^2}\phi(t, f)\Psi + \phi(t, (m^2 - \Delta)f)\Psi = (J_t, f).$$

Remark 2.1. *Let $\Psi \in D(N_{\mathrm{f}}^{1/2})$. Then $\mathcal{S}(\mathbb{R}^d) \ni f \mapsto (\Psi, \phi(t, f)\Psi)$ and $\mathscr{S}(\mathbb{R}^d) \ni g \mapsto (\Psi, \pi(t, g)\Psi)$ are tempered distributions and symbols $\varphi_\Psi(t, x)$ and $\varpi_\Psi(t, x)$, defined formally as*

$$\int \varphi_\Psi(t, x) f(x) dx = (\Psi, \pi(t, f)\Psi) \quad and \quad \int \varpi_\Psi(t, x) f(x) dx = (\Psi, \pi(t, f)\Psi),$$

satisfy
$$\varphi_\Psi(t, x) \in H^{1/2}(\mathbb{R}^3), \quad \varpi_\Psi(t, x) \in H^{-1/2}(\mathbb{R}^3).$$

We denote by Γ_0 the linear hull of

$$\{\Omega\} \cup \{c^*(f_1) \cdots c^*(f_n)\Omega \mid f_j \in D(\omega_0), \ j = 1, \cdots, n, \ n \geq 1\}.$$

Note that Γ_0 is dense in $\Gamma(\mathfrak{h})$ and that $\phi(t, f)$ and $\pi(t, f)$ $(f \in \mathcal{S}(\mathbb{R}^3))$ leave Γ_0 invariant. Suppose that $J_t \in H^{1/2}(\mathbb{R}^3)$ holds. If Ψ is a vector

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belonging to Γ_0, *then* $\varphi_\Psi(t) \in H^{3/2}(\mathbb{R}^3)$, $\varpi_\Psi(t) \in H^{1/2}(\mathbb{R}^3)$ *and*

$$\begin{bmatrix} \varphi_\Psi(t) \\ \varpi_\Psi(t) \end{bmatrix} = \begin{bmatrix} \cos(t\omega) & \omega^{-1}\sin(t\omega) \\ -\omega\sin(t\omega) & \cos(t\omega) \end{bmatrix} \begin{bmatrix} \varphi_\Psi(0) \\ \varpi_\Psi(0) \end{bmatrix}$$
$$+ \begin{bmatrix} \int_0^t ds \sin[(t-s)\omega]\omega^{-1} J_s \\ -\int_0^t ds \cos[(t-s)\omega] J_s \end{bmatrix},$$

which gives a solution of

$$i\frac{d}{dt} \begin{bmatrix} \varphi(t) \\ \varpi(t) \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \varpi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ J_t \end{bmatrix} \qquad (2.10)$$

with the initial value

$$\begin{bmatrix} \varphi(0) \\ \varpi(0) \end{bmatrix} = \begin{bmatrix} \varphi_\Psi(0) \\ \varpi_\Psi(0) \end{bmatrix}. \qquad (2.11)$$

We note that $\Psi \in \Gamma_\infty$ belongs to $D(N^{1/2})$ and is an analytic vector of $\phi(t, f)$ and $\pi(t, f)$, i.e., for any $0 < t \leq t_0$,

$$\sum_{n=0}^\infty \frac{t^n}{n!} \|\phi(t, f)^n \Psi\| < \infty, \quad \text{and} \quad \sum_{n=0}^\infty \frac{t^n}{n!} \|\pi(t, f)^n \Psi\| < \infty$$

with some $t_0 > 0$.

Remark 2.2. *Suppose that* $J_t \in H^{1/2}(\mathbb{R}^3)$ *holds. Then the pair of field operators* $\phi(t, f)$ *and* $\pi(t, f)$ *is unique in the following sense: If there exist a pair of field operators* $\phi'(t, f) = \mathscr{U}'(t)^*\phi_0(f)\mathscr{U}'(t)$ *and* $\pi'(t, f) = \mathscr{U}'(t)^*\pi_0(f)\mathscr{U}'(t)$ *with a family of unitary operators* $\mathscr{U}'(t)$ *satisfying the following conditions (1) - (4), then* $\phi(t, f) = \phi'(t, f)$ *and* $\pi(t, f) = \pi'(t, f)$:

(1) $\mathscr{U}'(0) = I$.

(2) $\phi'(t, f)$ *and* $\pi'(t, f)$ *leave* Γ_0 *invariant.*

(3) The vector $\Psi \in \Gamma_0$ *is an analytic vector of* $\phi'(t, f)$ *and* $\pi'(t, f)$.

(4) The distributional kernels $\varphi'_\Psi(t)$ *and* $\varpi'_\Psi(t)$ *of*

$$\varphi'_\Psi(t, f) = (\Psi, \phi'(t, f)\Psi),$$
$$\varpi'_\Psi(t, f) = (\Psi, \pi'(t, f)\Psi)$$

with $\Psi \in \Gamma_0$ *satisfy the equation (2.10) with the initial value (2.11).*

The uniqueness can be proved as follows. Using the conditions (1), (4) and the uniqueness of the solution of (2.10), we infer

$$\phi(t, f) = \phi'(t, f), \quad \pi(t, g) = \pi'(t, g)$$

on Γ_0. By the conditions (2) and (3), we have

$$e^{it\phi(t,f)}\Psi = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\phi(t, f)^n\Psi = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\phi'(t, f)^n\Psi = e^{i\phi'(t,f)}\Psi$$

for $\Psi \in \Gamma_0$ and sufficiently small $t > 0$. Since Γ_0 is dense, we observe that $e^{i\psi(t,f)} = e^{i\phi'(t,f)}$ for real $f \in H^{-1/2}(\mathbb{R}^3)$. By the uniqueness of the generator, we have the operator equality $\psi(t, f) = \phi'(t, f)$ for real $f \in H^{-1/2}(\mathbb{R}^3)$. Note that $\phi(t, f)$ and $\phi'(t, f)$ are unitary equivalent to $\phi_0(f)$. By the similar argument as in the proof of the closedness of $\phi_0(f)$, one can prove that the following operator equation holds for non real $f \in H^{-1/2}(\mathbb{R}^3)$:

$$\begin{aligned} \phi(t, f) &= \phi(t, \mathrm{Re}f) + i\phi(t, \mathrm{Im}f) \\ &= \phi'(t, \mathrm{Re}f) + i\phi'(t, \mathrm{Im}f) = \phi'(t, f) \end{aligned}$$

$\pi(t, f) = \pi'(t, f)$ is proved similarly.

2.4 Wave operators

Let

$$\mathscr{U}_0(t) = e^{-itH_\mathrm{f}}$$

and

$$U_0(t) = \begin{bmatrix} e^{-it\omega_0} & 0 \\ 0 & e^{it\omega_0} \end{bmatrix}.$$

One observe that

$$\mathscr{W}(t) = \mathscr{U}(t)^*\mathscr{U}_0(t)$$

satisfies

$$\mathscr{W}(t)e^{i\psi(v)}\mathscr{W}(t)^* = \mathscr{U}_t^* e^{i(\psi(U_0(t)v)+i(j_t, QU_0(t)v))}\mathscr{U}_t \qquad (2.12)$$

$$= e^{i\psi((U(t)^*U_0(t)v)+i(U_0(t)^*j_t, Qv))} \qquad (2.13)$$

for $v \in \mathcal{H}_C$. By the Stone theorem, we have

$$\mathscr{W}(t)\psi(v)\mathscr{W}(t)^* = \psi(U(t)^*U_0(t)v) + i(U_0(t)^*j_t, Qv). \qquad (2.14)$$

Lemma 2.6. *Suppose that Assumptions 1.1 and 1.2 hold. Then:*

$$\text{s-}\lim_{t\to\pm\infty} U(t)^* U_0(t) = W_\pm,$$

where

$$W_\pm = \begin{bmatrix} (W_\pm)_{++} & (W_\pm)_{+-} \\ (W_\pm)_{-+} & (W_\pm)_{--} \end{bmatrix}$$

with

$$
\begin{aligned}
(W_\pm)_{++} &= \overline{(W_\pm)_{--}} \\
&= \frac{1}{2}(\omega_0^{-1/2} w_\pm \omega_0^{+1/2} + \omega_0^{+1/2} w_\pm \omega_0^{-1/2}), \\
(W_\pm)_{-+} &= \overline{(W_\pm)_{+-}} \\
&= \frac{1}{2}(\omega_0^{-1/2} w_\pm \omega_0^{+1/2} - \omega_0^{+1/2} w_\pm \omega_0^{-1/2}).
\end{aligned}
$$

Proof. See Appendix A.3. □

Lemma 2.7. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\lim_{t\to\pm\infty} U_0(t)^* j_t = j_\pm,$$

where

$$j_\pm = \begin{bmatrix} g_\pm \\ \bar{g}_\pm \end{bmatrix} \quad \text{and} \quad g_\pm = -\frac{1}{\sqrt{2}} \int_0^{\pm\infty} ds\, \omega_0^{-1/2} e^{is\omega_0} w_\pm^* J_s.$$

Proof. It suffices to prove that

$$\lim_{t\to\pm\infty} e^{it\omega_0} g_t = -\frac{1}{\sqrt{2}} \int_0^{\pm\infty} ds\, \omega_0^{-1/2} e^{is\omega_0} w_\pm^* J_s. \qquad (2.15)$$

By a direct calculation, we have

$$
\begin{aligned}
e^{it\omega_0} g_t = -\frac{1}{2\sqrt{2}} \int_0^t ds\, e^{it\omega_0} \omega_0^{-1/2} (\omega - \omega_0) \omega_0^{-1/2} \cdot \omega_0^{1/2} \omega^{-1/2} \\
\times e^{i(t-s)\omega} \omega^{-1/2} J_s \\
-\frac{1}{2\sqrt{2}} \int_0^t ds\, e^{it\omega_0} \left(\omega_0^{-1/2} + \omega_0^{1/2} \omega^{-1}\right) e^{-i(t-s)\omega} J_s. \quad (2.16)
\end{aligned}
$$

Since $\omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2}$ is Hilbert-Schmidt by Lemma 2.2 and since $\int_0^t ds\|\omega^{-1/2} J_s\| < \infty$ by (b) of Assumption 1.3, the first term of the

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r.h.s in (2.16) tends to zero as t goes to $\pm\infty$. We show that the limit of the second term in (2.16) equals (2.15). It holds that

$$e^{it\omega_0}\left(\omega_0^{-1/2} + \omega_0^{1/2}\omega^{-1}\right)e^{-it\omega}$$
$$= \left(\omega_0^{-1/2}e^{it\omega_0}e^{-it\omega} + e^{it\omega_0}e^{-it\omega}\omega^{-1/2}\right)$$
$$+ e^{it\omega_0}(\omega_0^{-1/2} - \omega^{-1/2})\omega^{-1}e^{-it\omega}. \qquad (2.17)$$

The first term of the r.h.s in (2.17) tends to $2\omega_0^{-1/2}w_\pm^*$. The second term tends to zero since

$$(\omega_0^{-1/2} - \omega^{-1/2})\omega^{-1}e^{-it\omega} = \omega_0^{-1/2}(1 - \omega_0^{1/2}\omega^{-1/2})\omega^{-1}e^{-it\omega}$$

and since, by Lemma 2.2, $(1 - \omega_0^{1/2}\omega^{-1/2})$ is Hilbert-Schmidt. Using these facts and Assumption 1.3 (b) again, we infer the limit of the second term in (2.16) equals (2.15). $\qquad\square$

By Lemmas 2.6 and 2.7, we have

Lemma 2.8. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then it holds that, for $v \in \mathcal{H}_C$,*

$$s\text{-}\lim_{t\to\pm} \mathscr{W}(t)e^{i\psi(v)}\mathscr{W}(t)^* = e^{i(\psi(W_\pm v)+i(j_\pm, Qv))}. \qquad (2.18)$$

In particular, the following properties hold: for real $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$,

$$s\text{-}\lim_{t\to\pm} \mathscr{W}(t)e^{i\phi_0(f)}\mathscr{W}(t)^* = e^{i\phi_\pm(f)},$$
$$s\text{-}\lim_{t\to\pm} \mathscr{W}(t)e^{i\pi_0(g)}\mathscr{W}(t)^* = e^{i\pi_\pm(g)},$$

where

$$\phi_\pm(f) = \psi(W_\pm u_0) + i(j_\pm, Qu_0),$$
$$\pi_\pm(f) = \psi(W_\pm v_0) + i(j_\pm, Qv_0).$$

Proof. Since $e^{i\psi(v)}$ ($v \in \mathcal{H}_C$) is unitary, it suffices to prove (2.18) on D_f, which is an easy exercise. $\qquad\square$

Lemma 2.9. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then there exists a unitary operator $\mathscr{W}_\pm$ such that*

$$\mathscr{W}_\pm e^{i\psi(v)}\mathscr{W}_\pm^* = e^{i(\psi(W_\pm v)+i(j_\pm, Qv))}, \quad v \in \mathcal{H}_C.$$

In particular, it holds that: for $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$,

$$\mathscr{W}_\pm\phi_0(f)\mathscr{W}_\pm^* = \phi_\pm(f),$$
$$\mathscr{W}_\pm\pi_0(f)\mathscr{W}_\pm^* = \pi_\pm(f).$$

Proof. Note that $(W_\pm^*)_{-+} = W_{+-}^*$ is equal to

$$\frac{1}{2}\omega_0^{1/2} w_\mp^* \omega_0^{-1/2} - \omega_0^{-1/2} w_\mp^* \omega_0^{1/2})$$
$$= \frac{1}{2}\left(w_\mp^* \omega^{-1/2}\omega_0^{1/2}\right) \cdot \left(\omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2}\right)$$

and hence is Hilbert-Schmidt. By Lemma 2.1, it holds that there exists a unitary operator $\mathscr{U}_\pm$ such that

$$\mathscr{U}_\pm e^{i\psi(v)}\mathscr{U}_\pm^* = e^{i\psi(W_\pm v)}, \quad v \in \mathcal{H}_C.$$

Setting $\mathscr{W}_\pm = \mathscr{U}_\pm e^{i\psi(j_\pm)}$, one obtains the desired result. $\qquad\square$

3 Scattering theory

Throughout this section, we suppose that Assumptions 1.1 - 1.3 hold. Since any two quantum systems which are transformed from one to the other by a unitary transformation are physically equivalent, one can redefine the solution of the Klein-Gordon field as

$$\boldsymbol{\phi}(t,f) = \mathscr{W}_-^* \phi(t,f)\mathscr{W}_-, \quad \boldsymbol{\pi}(t,g) = \mathscr{W}_-^* \pi(t,g)\mathscr{W}_-$$

for $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$. Then it follows from Proposition 2.4 (see also Remarks 2.1 - 2.2) that $\boldsymbol{\phi}(t,f)$ and $\boldsymbol{\pi}(t,g)$ satisfy (1.1) in the operator valued distributional sense. The field operators $\boldsymbol{\phi}_s(t,f)$ and $\boldsymbol{\pi}_s(t,g)$ defined by

$$\boldsymbol{\phi}_s(t,f) = \mathscr{W}_-^* \mathscr{U}(s)^* \mathscr{U}(s-t)\phi_0(f)\mathscr{U}(s-t)^*\mathscr{U}(s)\mathscr{W}_-,$$
$$\boldsymbol{\pi}_s(t,g) = \mathscr{W}_-^* \mathscr{U}(s)^* \mathscr{U}(s-t)\pi_0(g)\mathscr{U}(s-t)^*\mathscr{U}(s)\mathscr{W}_-$$

satisfy the free Klein-Gordon equation with the initial condition:

$$\boldsymbol{\phi}_s(s,f) = \boldsymbol{\phi}(s,f), \quad \boldsymbol{\pi}_s(t,g) = \boldsymbol{\pi}(s,g).$$

The asymptotic fields $\boldsymbol{\phi}_\sharp(t,f)$ and $\boldsymbol{\pi}_\sharp(t,g)$ ($\sharp$ stands for out or in) are defined as

$$e^{i\boldsymbol{\phi}_{\mathrm{out/in}}(t,f)} = \lim_{t\to\pm\infty} e^{i\boldsymbol{\phi}_s(t,f)}, \quad e^{i\boldsymbol{\pi}_{\mathrm{out/in}}(t,g)} = \lim_{t\to\pm\infty} e^{i\boldsymbol{\pi}_s(t,g)}$$

for any real $f \in H^{-1/2}(\mathbb{R}^3)$ and $g \in H^{1/2}(\mathbb{R}^3)$. Then, by Lemmas 2.8 and 2.9, the incoming fields $\boldsymbol{\phi}_{\mathrm{in}}(f) = \boldsymbol{\phi}_{\mathrm{in}}(0,f)$ and $\boldsymbol{\pi}_{\mathrm{in}}(g) = \boldsymbol{\pi}_{\mathrm{in}}(0,g)$ are

$$\boldsymbol{\phi}_{\mathrm{in}}(f) = \phi_0(f), \quad \boldsymbol{\pi}_{\mathrm{in}}(g) = \pi_0(g) \tag{3.1}$$

and

$$\{\boldsymbol{\phi}_{\mathrm{in}}(f), \boldsymbol{\pi}_{\mathrm{in}}(f) \mid f \in H^{-1/2}(\mathbb{R}^3),\ g \in H^{1/2}(\mathbb{R}^3)\}$$

gives the Fock representation of the CCR (see, e.g., [1]). The out going fields $\phi_{\text{out}}(f) = \phi_{\text{out}}(0, f)$ and $\pi_{\text{out}}(g) = \pi_{\text{out}}(0, g)$ are

$$\phi_{\text{out}}(f) = (\mathscr{W}_+^* \mathscr{W}_-)^* \phi_0(f)(\mathscr{W}_+^* \mathscr{W}_-), \quad \pi_{\text{in}}(g) = (\mathscr{W}_+^* \mathscr{W}_-)^* \pi_0(g)(\mathscr{W}_+^* \mathscr{W}_-).$$
$$(3.2)$$

The scattering matrix $\mathscr{S} = \mathscr{S}(V, J)$ is defined by

$$\mathscr{S} = \mathscr{W}_+^* \mathscr{W}_-.$$

Proposition 3.1. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\mathscr{S}^{-1}\phi_{\text{in}}(f)\mathscr{S} = \phi_{\text{out}}(f) \quad \text{and} \quad \mathscr{S}^{-1}\pi_{\text{in}}(f)\mathscr{S} = \pi_{\text{out}}(f).$$

Proof. By (3.1) and (3.2), we see that

$$\mathscr{S}^{-1}\psi_{\text{in}}(f)\mathscr{S} = \mathscr{W}_-^* \mathscr{W}_+ \psi_0(f)\mathscr{W}_+^* \mathscr{W}_- = \psi_{\text{out}}(f),$$

where $\psi_{\text{in/out},0}$ stands for $\phi_{\text{in/out},0}$ or $\pi_{\text{in/out},0}$. $\square$

Let us define the associated annihilation and creation operators by

$$c_{\text{in}}(f) = c(f) \quad \text{and} \quad c_{\text{out}}(f) = \mathscr{S}^* c(f)\mathscr{S}$$

and $c_\sharp^*(f) = c_\sharp(f)^*$. The free Hamiltonian of the incoming field and outgoing field are defined by

$$H_{\text{in}} = d\Gamma(\omega_0) \quad \text{and} \quad H_{\text{out}} = \mathscr{S}^* d\Gamma(\omega_0)\mathscr{S}.$$

The asymptotic vacua

$$\Omega_{\text{in}} = \Omega \quad \text{and} \quad \Omega_{\text{out}} = \mathscr{S}^*\Omega$$

satisfy

$$H_\sharp \Omega_\sharp = 0 \quad \text{and} \quad c_\sharp(f)\Omega_\sharp = 0.$$

The asymptotic fields satisfy

$$\phi_\sharp(t, f) = e^{itH_\sharp}\phi_\sharp(f)e^{-itH_\sharp}, \quad \text{and} \quad \pi_\sharp(t, f) = e^{itH_\sharp}\pi_\sharp(f)e^{-itH_\sharp},$$

and

$$\phi_\sharp(t, f) = \frac{1}{\sqrt{2}} \left(c_\sharp^*(e^{it\omega_0}\omega_0^{-1/2}f) + c_\sharp(e^{it\omega_0}\omega_0^{-1/2}\bar{f}) \right),$$
$$\pi_\sharp(t, f) = \frac{i}{\sqrt{2}} \left(c_\sharp^*(e^{it\omega_0}\omega_0^{+1/2}f) - c_\sharp(e^{it\omega_0}\omega_0^{+1/2}\bar{f}) \right)$$

hold on $\tilde{D}$.

4 Inverse scattering

4.1 Uniqueness of the potential V

Let

$$S = w_+^* w_-.$$

When we want to emphasize the dependence of V, we write $S = S(V)$. We will prove the following theorem:

Theorem 4.1. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. If $\mathscr{S}(V, J) = \mathscr{S}(V', J')$, then $S(V) = S(V')$.*

We need the following:

Lemma 4.2. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\lim_{t \to \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathscr{S}c^*(e^{it\omega_0} g)\Omega) = (\Omega, \mathscr{S}\Omega)(f, Sg). \qquad (4.1)$$

Proof of Theorem 4.1. For notational simplicity, we write $\mathscr{S}(V', J') = \mathscr{S}'$ and $S(V') = S'$. Note that $(\Omega, \mathscr{S}\Omega) \neq 0$ since $\mathscr{S}$ is unitary. By Lemma 4.2, we have

$$
\begin{aligned}
(f, Sg) &= \lim_{t \to \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathscr{S}c^*(e^{it\omega_0} g)\Omega)/(\Omega, \mathscr{S}\Omega) \\
&= \lim_{t \to \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathscr{S}'c^*(e^{it\omega_0} g)\Omega)/(\Omega, \mathscr{S}'\Omega) \quad = (f, S'g).
\end{aligned}
$$

$\square$

In the remainder of this subsection, we will prove Lemma 4.2. Let

$$\mathscr{I}_t(f, g) = (c^*(e^{it\omega_0} f)\Omega, \mathscr{S}c^*(e^{it\omega_0} g)\Omega).$$

We see from Lemma 2.9 that $\mathscr{W}_\pm = \mathscr{U}_\pm e^{i\psi(j_\pm)}$. By a direct calculation, one obtains the following:

$$
\begin{aligned}
\mathscr{U}_\pm c(f)\mathscr{U}_\pm^* &= c((W_\pm)_{++} f) + c^*((W_\pm)_{+-} \bar{f}), \\
\mathscr{U}_\pm c^*(f)\mathscr{U}_\pm^* &= c^*((W_\pm)_{++} f) + c((W_\pm)_{+-} \bar{f}), \\
e^{i\psi(j_\pm)} c(f) e^{-i\psi(j_\pm)} &= c(f) - i(f, j_\pm), \\
e^{i\psi(j_\pm)} c^*(f) e^{-i\psi(j_\pm)} &= c^*(f) + i(j_\pm, f).
\end{aligned}
$$

It holds from the above that

$$
\begin{aligned}
&\mathscr{U}_\pm e^{i\psi(j_\pm)} c^* \left(e^{it\omega_0} f\right) \Omega \\
&= \left(c^* \left((W_\pm)_{++} e^{it\omega_0} f\right) + c \left((W_\pm)_{+-} e^{-it\omega_0} \bar{f}\right) + i(g_\pm, e^{it\omega_0} f)\right) \mathscr{U}_\pm e^{i\psi(j_\pm)}\Omega \\
&= c^* \left((W_\pm)_{++} e^{it\omega_0} f\right) \mathscr{U}_\pm e^{i\psi(j_\pm)}\Omega + o(1)
\end{aligned}
$$

as $t \to \pm\infty$ in the strong topology since $(W_\pm)_{+-}$ is Hilbert-Schmidt and since $\lim_{t\to\pm\infty}(g_\pm, e^{it\omega_0}f) = 0$ by the Riemann-Lebesgue Lemma. Hence we have

$$
\begin{aligned}
&\mathscr{I}_t(f,g) \\
&= \left(c^*\left((W_+)_{++}e^{it\omega_0}f\right)\mathscr{U}_+e^{i\psi(j_+)}\Omega, c^*\left((W_-)_{++}e^{it\omega_0}g\right)\mathscr{U}_-e^{i\psi(j_-)}\Omega\right) + o(1) \\
&= \left((W_+)_{++}e^{it\omega_0}f, (W_-)_{++}e^{it\omega_0}g\right)(\Omega, \mathscr{S}\Omega) \\
&\quad + \left(c\left((W_-)_{++}e^{it\omega_0}g\right)\mathscr{U}_+e^{i\psi(j_+)}\Omega, c\left((W_+)_{++}e^{it\omega_0}f\right)\mathscr{U}_-e^{i\psi(j_-)}\Omega\right) + o(1)
\end{aligned}
\tag{4.2}
$$

as $t \to \pm\infty$. It is straightforward that

$$
\begin{aligned}
&(W_+)^*_{++}(W_-)_{++} \\
&= \frac{1}{4}\left(\omega_0^{-1/2}w_+^*\omega_0^{1/2} + \omega_0^{+1/2}w_+^*\omega_0^{-1/2}\right)\left(\omega_0^{-1/2}w_-\omega_0^{1/2} + \omega_0^{+1/2}w_-\omega_0^{-1/2}\right) \\
&= S + \frac{1}{4}w_+^*\left[(\omega^{-1/2}\omega_0\omega^{-1/2} - 1) + (\omega^{1/2}\omega_0^{-1}\omega^{1/2} - 1)\right]w_-.
\end{aligned}
\tag{4.3}
$$

Note that $Se^{it\omega_0} = e^{it\omega_0}S$ and that the second term in the r.h.s of (4.3) is Hilbert-Schmidt by Lemma 2.2. Hence, by (4.2), we have (4.1) if the following holds true:

$$
\left(c\left((W_-)_{++}e^{it\omega_0}g\right)\mathscr{U}_+e^{i\psi(j_+)}\Omega, c\left((W_+)_{++}e^{it\omega_0}f\right)\mathscr{U}_-e^{i\psi(j_-)}\Omega\right) = o(1).
\tag{4.4}
$$

To prove (4.4), we use the relation

$$
\mathscr{U}_\pm^*c(f)\mathscr{U}_\pm = c\left((W_\pm)^*_{++}\right) - c^*((W_\pm)^*_{-+}\bar{f}),
$$

which is obtained from $\mathscr{U}_\pm^*e^{i\psi(v)}\mathscr{U}_\pm = e^{i\psi(QW_\pm^*Qv)}$. By the above and the fact that $(W_\pm)^*_{-+}$ is Hilbert-Schmidt, we observe that

$$
c\left((W_-)_{++}e^{it\omega_0}g\right)\mathscr{U}_+e^{i\psi(j_+)}\Omega = \mathscr{U}_+e^{i\psi(j_+)}c\left((W_-)_{++}e^{it\omega_0}g\right)\Omega + o(1).
$$

Since the first term of the above equals zero, the proof of the lemma is completed.

4.2 Characterization of the external source J

Our aim of this subsection is to represent the classical source J in terms of the functional

$$
F(t,f) = (\Omega_{\mathrm{in}}, \phi_{\mathrm{out}}(t,f)\Omega_{\mathrm{in}}), \quad f \in \mathcal{S}(\mathbb{R}^d).
$$

We set

$$Z_\pm[f] = X[f] \mp iY[f],$$

where

$$X[f] = \frac{d}{dt}F(t, \omega_0^{-1/2}f)\Big|_{t=0} \quad \text{and} \quad Y[f] = F(0, \omega_0^{+1/2}f).$$

Note that

$$\phi_0(t, f) = \psi(v_t),$$

where v_t is

$$v_t = \begin{bmatrix} e^{it\omega_0}\omega_0^{-1/2}\bar{f}/\sqrt{2} \\ e^{-it\omega_0}\omega_0^{-1/2}\bar{f}/\sqrt{2} \end{bmatrix}.$$

We see that

$$Cv_t = \begin{bmatrix} e^{it\omega_0}\omega_0^{-1/2}f/\sqrt{2} \\ e^{-it\omega_0}\omega_0^{-1/2}f/\sqrt{2} \end{bmatrix} \neq v_t$$

and $v_t \notin \mathcal{H}_C$. By a direct calculation, we have for $v \in \mathcal{H}_C$:

$$\mathscr{S}^{-1}\psi(v)\mathscr{S} = \psi(QW_-^*QW_+v) + i(Qj_+ - W_+^*QW_-j_-, v).$$

Since $(\Omega, \psi(v)\Omega) = 0$ and

$$\begin{aligned} (Qj_+ - W_+^*QW_-j_-, v) &= (W_+^*Q(W_+Qj_+ - W_-j_-), v) \\ &= (W_+j_+ - W_-j_-, QW_+v), \end{aligned}$$

one obtains

$$(\Omega, \mathscr{S}^{-1}\psi(v)\mathscr{S}\Omega) = i(W_+j_+ - W_-j_-, QW_+v), \quad v \in \mathcal{H}_C.$$

It holds that

$$\begin{aligned} F(t, f) &= \frac{1}{2}(\Omega, \mathscr{S}^{-1}\psi(v_t + Cv_t)\mathscr{S}\Omega) \\ &\quad + \frac{i}{2}(\Omega, \mathscr{S}^{-1}\psi(i(v_t - Cv_t))\mathscr{S}\Omega) \\ &= \frac{i}{2}(W_+j_+ - W_-j_-, QW_+(v_t + Cv_t)) \\ &\quad - \frac{i}{2}(W_+j_+ - Q_-j_-, QW_+(v_t - Cv_t)) \\ &= i(W_+j_+ - W_-j_-, QW_+Cv_t). \end{aligned}$$

Thus we infer

$$X[f] = \frac{1}{\sqrt{2}}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} f \\ -f \end{bmatrix}\right),$$

$$Y[f] = \frac{i}{\sqrt{2}}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} f \\ f \end{bmatrix}\right)$$

and

$$Z_+[f] = \sqrt{2}\left(W_+ j_+ - W_- j_-, QW_+ \begin{bmatrix} f \\ 0 \end{bmatrix}\right),$$

$$Z_-[f] = -\sqrt{2}\left(W_+ j_+ - W_- j_-, QW_+ \begin{bmatrix} 0 \\ f \end{bmatrix}\right).$$

Lemma 4.3. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Let*

$$Z[f,g] := Z_+[f] - Z_-[g].$$

Then

$$Z[Sf, f] = -2\sqrt{2}i(\mathrm{Im}(g_\infty), w_- f),$$
$$Z[Sf, -f] = 2\sqrt{2}(\mathrm{Re}(g_\infty), w_- f),$$

where

$$g_\infty := w_+ g_+ - w_- g_- = -\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} ds\, \omega^{-1/2} e^{is\omega} J_s.$$

Proof. By definition, we see that

$$Z[w_+^* f, \pm w_-^* f] = \sqrt{2}\left(W_+ j_+ - W_- j_-, QW_+ \begin{bmatrix} w_+^* f \\ \pm w_-^* f \end{bmatrix}\right).$$

By direct calculation, we have

$$W_+ j_+ - QW_- j_-$$
$$= \begin{bmatrix} \omega_0^{-1/2}\omega^{1/2}\mathrm{Re}(g_\infty) + i\omega_0^{1/2}\omega^{-1/2}\mathrm{Im}(g_\infty) \\ \omega_0^{-1/2}\omega^{1/2}\mathrm{Re}(g_\infty) - i\omega_0^{1/2}\omega^{-1/2}\mathrm{Im}(g_\infty) \end{bmatrix}$$

and

$$QW_+ \begin{bmatrix} w_+^* f \\ \pm w_-^* f \end{bmatrix} = \begin{bmatrix} \omega_0^{\mp 1/2}\omega^{\pm 1/2} f \\ \mp \omega_0^{\mp 1/2}\omega^{\pm 1/2} f \end{bmatrix},$$

which completes the proof. $\qquad\square$

We introduce the functional $Z : L^2(\mathbb{R}^3) \to \mathbb{C}$ by

$$Z[f] := \frac{1}{2\sqrt{2}}\left(Z[Sf, f] + Z[Sf, -f]\right), \quad f \in L^2(\mathbb{R}^3).$$

For $f \in L^2(\mathbb{R}^3)$, $\lambda > 0$ and $k, x \in \mathbb{R}^3$, we denote $e^{-ik\cdot x}f(\lambda x)$ by $f_k^\lambda(x)$. $\mathcal{F}_0$ stands for the Fourier transform: $\mathfrak{h} \ni f \mapsto (\mathcal{F}_0 f) = \hat{f}$ and $\hat{f}(k) = (2\pi)^{-3/2}\int dk\, e^{-ik\cdot x}f(x)$. The generalized Fourier transform $\mathcal{F}_\pm$ is defined by $\mathcal{F}_0 w_\pm^*$. Let $\chi \in \mathfrak{h}$ such that $0 \le \chi \le 1$ and $\chi(x) = 1$ if $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$. We introduce a function z_λ by

$$z_\lambda(k) := -\sqrt{2}(2\pi)^{-3/2} Z[\chi_k^\lambda].$$

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Lemma 4.4. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then $z_\lambda \in \mathfrak{h}$ and*

$$\lim_{\lambda \to 0} z_\lambda = \int_{-\infty}^{\infty} ds \mathcal{F}_+ \left(e^{-is\omega} \omega^{-1/2} J_s \right) \quad in \ \mathfrak{h}.$$

Proof. Since, by the above lemma, we have

$$Z[f] = (w_-^* g_\infty, f), \tag{4.5}$$

it follows that

$$z_\lambda(k) = \left(\int_{-\infty}^{\infty} ds w_-^* e^{is\omega} \omega^{-1/2} J_s, \chi_k^\lambda \right)$$

$$= (2\pi)^{-3/2} \int dx e^{-ik \cdot x} \chi(\lambda x) \times \left[\int_{-\infty}^{\infty} ds w_+^* e^{is\omega} \omega^{-1/2} J_s \right](k).$$

Hence z_λ converges in $\mathfrak{h}$ to

$$\mathcal{F}_0 \left[\int_{-\infty}^{\infty} ds w_+^* e^{-is\omega} \omega^{-1/2} J_s \right] = \int_{-\infty}^{\infty} ds \mathcal{F}_+ e^{-is\omega} \omega^{-1/2} J_s.$$

$\square$

Let

$$z := \lim_{\lambda \to 0} z_\lambda. \tag{4.6}$$

Proposition 4.5. *Suppose that Assumptions 1.1 - 1.2 and 1.3 hold. If $\mathscr{S}(V, J) = \mathscr{S}(V', J')$, then*

$$\int_{-\infty}^{\infty} ds e^{-is\omega} J_s = \int_{-\infty}^{\infty} ds e^{-is\omega} J_s'.$$

Proof. Let $z'(k)$ be defined as $z(k)$ with replacing $\mathscr{S}(V, J)$ by $\mathscr{S}(V', J')$. By Theorem 4.1, we have $S(V) = S(V')$, $V = V'$ and hence $z = z'$. By Lemma 4.4, we obtain

$$\int_{-\infty}^{\infty} ds e^{-is\omega} \omega^{-1/2} J_s = \int_{-\infty}^{\infty} ds e^{-is\omega} \omega^{-1/2} J_s'$$

by the unitarity of $\mathcal{F}_+$. $\square$

Henceforth, we suppose that J is expressed by

$$J(t, x) = j(t)\rho(x), \tag{4.7}$$

where $j \in L^1(\mathbb{R})$ and $\rho \in H^{-1/2}(\mathbb{R}^3)$. In Subsection 4.3 below, assuming that j is a given function and that $\hat{j}$ is analytic, we will represent ρ in terms of z and j. In Subsection 4.4 below, we next assume that ρ is a given function. We will show that j is determined by z and ρ if j satisfies the following:

$$\text{For some } \delta > 0, \quad e^{\delta|t|} j(t) \in L^1(\mathbb{R}_t). \tag{4.8}$$

4.3 Reconstruction of ρ

Let j be a given function belonging to $L^1(\mathbb{R}_t)$. Then we immediately obtain the reconstruction formula for determining ρ:

Theorem 4.6. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Assume that $J \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^3)$ satisfies (4.7) and that $\hat{j}$ is a nonzero, real analytic given function. If $\omega(k) \notin (\hat{j})^{-1}(0)$, then $(\mathcal{F}_+\rho)(k)$ is uniquely determined by*

$$(\mathcal{F}_+\rho)(k) = \frac{z(k)}{(\hat{j})(\hat{\omega}_0(k))}. \tag{4.9}$$

Remark 4.1. *Since $\hat{j}$ is real analytic function, $(\hat{j})^{-1}(0)$ is a discrete set and hence countable. Thus, (4.9) holds for almost all $k \in \mathbb{R}^3$ and we have $\rho = \mathcal{F}_+(z/(\hat{j}(\hat{\omega}_0)))$. If the generalized eigenfunction $\psi_+(k,x)$ of $-\Delta + V$ exists, then the inverse of the generalized Fourier transform $\mathcal{F}_+$ is given by*

$$(\mathcal{F}_+^{-1}f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} dk \, \psi_+(k,x) f(k),$$

where we denote $\underset{R \to +\infty}{\mathrm{l.i.m.}} \int_{|k| \le R} dk$ by $\int_{\mathbb{R}^3} dk$. In this case, we have

$$\rho(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} dk \frac{\psi_+(k,x) z(k)}{(\hat{j})(\hat{\omega}_0(k))}.$$

4.4 Reconstruction of j

In this subsection we suppose that ρ is a nonzero, given function belonging to $H^{-1/2}(\mathbb{R}^3)$. The following lemma will help us to identify j:

Lemma 4.7. *Suppose that j satisfies (4.8). Then $\hat{j}$ is real analytic. Furthermore, the radius of convergence of the Taylor series of $\hat{j}$ is larger than δ.*

Proof. By the assumption (4.8), it follows that for any non negative integer m,

$$|t|^m j(t) \in L^1(\mathbb{R}_t).$$

Therefore, we have $\hat{j} \in C^\infty(\mathbb{R})$ and

$$(\hat{j})^{(m)}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-it)^m e^{-i\tau t} j(t) dt, \quad \tau \in \mathbb{R},$$

where $(\hat{j})^{(m)}$ is the m-th order derivative of $\hat{j}$.

Since

$$\sup_{t>0} t^m e^{-\delta t} = m^m \delta^{-m} e^{-m},$$

we obtain for any $\tau \in \mathbb{R}$,

$$\begin{aligned}
\left| (\hat{j})^{(m)}(\tau) \right| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t|^m |j(t)| dt \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |j(t)| e^{\delta|t|} |t|^m e^{-\delta|t|} dt \\
&\leq \frac{1}{\sqrt{2\pi}} m^m \delta^{-m} e^{-m} \int_{\mathbb{R}} |j(t)| e^{\delta|t|} dt.
\end{aligned}$$

Thus, we have

$$\left| \frac{\widehat{(j)}^{(m)}(\tau)}{m!} \right| \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |j(t)| e^{\delta|t|} dt \right) \frac{m^m e^{-m}}{m!} \delta^{-m}.$$

Using Stirling's formula

$$m! = \sqrt{2\pi} m^{m+1/2} e^{-m} e^{\theta(m)/12m}, \quad 0 < \theta(m) < 1,$$

we see that

$$\limsup_{m \to \infty} \left| \frac{(\hat{j})^{(m)}(\tau)}{m!} \right|^{1/m} \leq \delta^{-1},$$

which completes the lemma. $\qquad \square$

Applying Lemmas 4.4 and 4.7, we have the following result:

Theorem 4.8. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Assume that J satisfies (4.7) and $\rho \in H^{-1/2}(\mathbb{R}^3)$ is a nonzero, given function. If j satisfies (4.8), then j is uniquely reconstructed by the following steps:*

(Step I) *Fix a point $k_0 \notin (\mathcal{F}_+\rho)^{-1}(0)$. Let U_0 be a k_0-neighborhood such that $0 \notin (\widehat{J_X})(U_0)$. Then we have*

$$(\hat{j})(\omega(k)) = \frac{z(k)}{(\mathcal{F}_+\rho)(k)}, \quad k \in U_0. \tag{4.10}$$

Therefore, we see exact values of $(\hat{j})^{(m)}(\tau_0)$, $m = 0, 1, 2, \cdots$, where $\tau_0 = \omega(k_0)$.

(Step II) For any $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$, we have

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0)}{m!}(\tau - \tau_0)^m.$$

(Step III) Let l be a positive integer. Suppose that we have already determined $(\hat{j})(\tau)$ with $\tau \in (\tau_0 - \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+1)\delta}{2})$. For any $\tau \in [\tau_0 + \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+2)\delta}{2})$, we see the value of $(\widehat{J_T})(\tau)$ by

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0 + l\delta/2)}{m!}(\tau - \tau_0 - l\delta/2)^m.$$

On the other hand, For any $\tau \in (\tau_0 - \frac{(l+2)\delta}{2}, \tau_0 - \frac{(l+1)\delta}{2}]$, we see the value of $(\widehat{J_T})(\tau)$ by

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0 - l\delta/2)}{m!}(\tau - \tau_0 + l\delta/2)^m.$$

(Step IV) From (Step III), $\hat{j}$ is reconstructed completely. Hence we can determine j by the inverse Fourier transform.

A   Appendix

A.1   Hilbert-Schmidt operators

We prove Lemma 2.2. We first show that $\omega^{1/2}\omega_0^{-1/2} - 1$. To this end, we use the formula

$$A^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty d\lambda \lambda^{\alpha-1}(A + \lambda)^{-1}A \qquad (A.1)$$

on $D(A)$ $(0 < \alpha < 1)$. With the aid of (A.1) for $\alpha = 1/4$, we have

$$\begin{aligned}
\omega^{1/2}\omega_0^{-1/2} - 1 &= [(\omega^2)^{1/4} - (\omega_0^2)^{1/4}]\omega_0^{-1/2} \\
&= \frac{1}{\sqrt{2}\pi}\int_0^\infty d\lambda\lambda^{-3/4}[(\omega^2 + \lambda)^{-1}\omega^2 - (\omega_0^2 + \lambda)^{-1}\omega_0^2]\omega_0^{-1/2} \\
&= \frac{1}{\sqrt{2}\pi}\int_0^\infty d\lambda\lambda^{1/4}[-(\omega^2 + \lambda)^{-1} + (\omega_0^2 + \lambda)^{-1}]\omega_0^{-1/2} \\
&= \frac{1}{\sqrt{2}\pi}\int_0^\infty d\lambda\lambda^{1/4}(\omega^2 + \lambda)^{-1}V(\omega_0^2 + \lambda)^{-1}\omega_0^{-1/2},
\end{aligned}$$

where the above equation holds true on some dense domain, e.g. $D(\omega_0^{3/2})$. It suffices to prove that the operator $(\omega^2 + \lambda)^{-1}V(\omega_0^2 +$

$\lambda)^{-1}\omega_0^{-1/2}$ is Hilbert-Schmidt and satisfy

$$\int_0^\infty d\lambda \lambda^{1/4}\|(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\omega_0^{-1/2}\|_2 < \infty, \qquad (A.2)$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Since $V \in L^2(\mathbb{R}^3)$ and $(\sqrt{|k|^2+m^2})^{-3/2-\epsilon} \in L^2(\mathbb{R}^3)$, $V\omega_0^{-3/2-\epsilon}$ is Hilbert-Schmidt for $\epsilon > 0$ (see [12, Theorem XI.20] for details). Since $\|(\omega^2+\lambda)^{-1}\| \le (m^2+\lambda)^{-1}$, we have

$\|(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\omega_0^{-1/2}\|_2$
$\le (m^2+\lambda)^{-1}\|V\omega_0^{-3/2-\epsilon}\|_2 \cdot \|\omega_0^{1+\epsilon}(\omega_0^2+\lambda)^{-1}\|$
$\le (m^2+\lambda)^{-1}\|V\omega_0^{-3/2-\epsilon}\|_2 \cdot \|\omega_0^{1+\epsilon}(\omega_0^2+\lambda)^{-(1+\epsilon)/2}\| \cdot \|(\omega_0^2+\lambda)^{-1+(1+\epsilon)/2}\|$
$\le (m^2+\lambda)^{-3/2+\epsilon/2}$

Thus (A.2) holds if $0 < \epsilon < 1/2$ and hence $\omega^{1/2}\omega_0^{-1/2} - 1$ is Hilbert-Schmidt.

We shall prove that $\omega^{-1/2}\omega_0^{1/2} - 1$ is Hilbert-Schmidt. From a similar argument as above, we infer that it suffices to show that

$$\int_0^\infty \lambda^{1/4}\|\omega^{-1/2}(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\|_2 < \infty. \qquad (A.3)$$

Since, as will be seen later, $\omega_0^{-1/2}V\omega_0^{-1-\epsilon}$ ($\epsilon > 0$) is Hilbert-Schmidt, we have

$\|\omega^{-1/2}(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\|_2 \le C(m^2+\lambda)^{-3/2+\epsilon/2}\|\omega_0^{-1/2}V\omega_0^{-1-\epsilon}\|_2$

with some positive C. Thus (A.4) holds true if $0 < \epsilon < 1/2$.

To prove that $\omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2}$ is Hilbert-Schmidt, we use the formula (A.1) for $\alpha = 1/2$ and write

$$\omega_0^{-1/2}(\omega_0-\omega)\omega_0^{-1/2} = -\frac{1}{\pi}\int_0^\infty d\lambda \lambda^{1/2}\omega_0^{-1/2}(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\omega_0^{-1/2}.$$

where the above equation hold true on some dense domain (for instance $D(\omega_0^{3/2})$). It suffices to show

$$\int_0^\infty d\lambda \lambda^{1/2}\|\omega_0^{-1/2}(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\omega_0^{-1/2}\|_2 < \infty. \qquad (A.4)$$

If $\omega_0^{-1/2}V\omega_0^{-1-\epsilon}$ ($\epsilon > 0$) is Hilbert-Schmidt, then

$\|\omega_0^{-1/2}(\omega^2+\lambda)^{-1}V(\omega_0^2+\lambda)^{-1}\omega_0^{-1/2}\|_2$
$\le \|\omega_0^{-1/2}(\omega^2+\lambda)^{-1}\omega_0^{1/2}\| \cdot \|\omega_0^{-1/2}V\omega_0^{-1-\epsilon}\|_2 \cdot \|\omega_0^{1/2+\epsilon}(\omega_0^2+\lambda)^{-1}\|$
$\le \|\omega_0^{-1/2}\omega^{1/2}\| \cdot \|(\omega^2+\lambda)^{-1}\| \cdot \|\omega^{-1/2}\omega_0^{1/2}\|$
$\quad \times \|\omega_0^{-1/2}V\omega_0^{-1-\epsilon}\|_2 \cdot \|\omega^{1/2+\epsilon}(\omega_0^2+\lambda)^{-(1/2+\epsilon)/2}\| \cdot \|(\omega_0^2+\lambda)^{-3/4+\epsilon/2}\|$
$\le C(m^2+\lambda)^{\epsilon/2-7/4}$

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with some $C > 0$ and $0 < \epsilon < 3/2$. This implies that (A.3) holds true for $0 < \epsilon < 1/2$ and we obtain the desired results.

It remains to show the following.

Lemma A.1. *For any $\epsilon > 0$, $\omega_0^{-1/2} V \omega_0^{-1-\epsilon}$ is Hilbert-Schmidt.*

Proof. Note that

$$\omega_0^{-1/2} V \omega_0^{-1-\epsilon} = V \omega_0^{-3/2-\epsilon} + [\omega_0^{-1/2}, V] \omega_0^{-1-\epsilon}.$$

As seen above, the first term of the r.h.s. is Hilbert-Schmidt. It suffices to show that $[\omega_0^{-1/2}, V]$ is Hilbert-Schmidt. To this end, we write it as

$$[\omega_0^{-1/2}, V] = \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2} [(\omega_0^2 + \lambda)^{-1}, V].$$

By direct calculation, we see that

$$[(\omega_0^2 + \lambda)^{-1}, V] = (\omega_0^2 + \lambda)^{-1} [V, \omega_0^2] (\omega_0^2 + \lambda)^{-1}$$
$$= V_1 + V_2,$$

where

$$V_1 = (\omega_0^2 + \lambda)^{-1} (\Delta V)(\omega_0^2 + \lambda)^{-1},$$
$$V_2 = 2(\omega_0^2 + \lambda)^{-1} (\nabla V) \cdot \nabla (\omega_0^2 + \lambda)^{-1}.$$

One observes that the operators $(\Delta V)(\omega_0^2 + \lambda)^{-1}$ and $(\omega_0^2 + \lambda)^{-1}(\nabla_j V)$ are integral operators with the kernels $(4\pi|x-y|)^{-1}(\Delta V)(x) e^{\sqrt{m^2+\lambda}|x-y|}$ and $(4\pi|x-y|)^{-1} e^{\sqrt{m^2+\lambda}|x-y|}(\nabla_j V)(y)$, respectively. Hence we have

$$\|(\Delta V)(\omega_0^2 + \lambda)^{-1}\|_2^2 = (4\pi)^{-2} \int dxdy \frac{|(\Delta V)(x)|^2 e^{-2\sqrt{m^2+\lambda}|x-y|}}{|x-y|^2}$$
$$= (4\pi)^{-2} \|(\Delta V)\|_{L^2}^2 \int dx \frac{e^{-2\sqrt{m^2+\lambda}|x|}}{|x|^2}$$
$$= \frac{1}{\sqrt{m^2+\lambda}} \left(\frac{\|(\Delta V)\|_{L^2}^2}{(4\pi)^2} \int dx \frac{e^{-2|x|}}{|x|^2} \right)$$

and

$$\|(\omega_0^2 + \lambda)^{-1}(\nabla_j V)\|_2^2 = \frac{1}{\sqrt{m^2+\lambda}} \left(\frac{\|(\nabla V)\|_{L^2}^2}{(4\pi)^2} \int dx \frac{e^{-2|x|}}{|x|^2} \right).$$

Thus we obtain

$$\|V_1\|_2 \leq \|(\omega_0^2 + \lambda)^{-1}\| \cdot \|V(\omega_0^2 + \lambda)^{-1}\|_2$$
$$\leq C(m^2 + \lambda)^{-5/4}$$

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and

$$\|V_2\|_2 \leq \sum_{j=1}^{3} \|(\omega_0^2 + \lambda)^{-1}(\nabla_j V)\|_2 \cdot \|\nabla_j(\omega_0^2 + \lambda)^{-1/2}\| \cdot \|(\omega_0^2 + \lambda)^{-1/2}\|$$

$$\leq C(m^2 + \lambda)^{-3/4}$$

with some C independent of λ. Hence $[\omega_0^{-1/2}, V]$ is Hilbert-Schmidt since

$$\|[\omega_0^{-1/2}, V]\|_2 \leq C \int_0^\infty d\lambda \lambda^{-1/2}((m^2 + \lambda)^{-5/4} + (m^2 + \lambda)^{-3/4}) < \infty.$$

$\square$

A.2 Existence of the limits (1.8)

Under Assumptions 1.1 and 1.2, we prove the existence of

$$\text{s-} \lim_{t \to \pm\infty} e^{it\omega} e^{-it\omega_0}.$$

Since $\omega - \omega_0$ is ω_0-bounded and $\mathcal{S}(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3) = \mathcal{H}_{ac}(\omega_0)$, it suffices to show that

$$\int_1^\infty dt \left\| (\omega - \omega_0)e^{-it\omega_0} f \right\|_{\mathfrak{h}} < \infty \tag{A.5}$$

for any $f \in \mathcal{S}(\mathbb{R}^3)$. From (A.1), we see that

$$\omega - \omega_0 = \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2}[(\omega^2 + \lambda)^{-1}\omega^2 - (\omega_0^2 + \lambda)^{-1}\omega_0^2]$$

$$= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2}[1 - \lambda(\omega^2 + \lambda)^{-1} - 1 + \lambda(\omega_0^2 + \lambda)^{-1}]$$

$$= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2}[(\omega^2 + \lambda)^{-1} - (\omega_0^2 + \lambda)^{-1}]$$

$$= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2}(\omega^2 + \lambda)^{-1}[(\omega_0^2 + \lambda) - (\omega^2 + \lambda)](\omega_0^2 + \lambda)^{-1}$$

$$= -\frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2}(\omega^2 + \lambda)^{-1}V(\omega_0^2 + \lambda)^{-1}.$$

Therefore, we have for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\|(\omega - \omega_0)e^{-it\omega_0} f\|_{\mathfrak{h}}$$

$$\leq C \int_0^\infty d\lambda \lambda^{1/2} \left\| (\omega^2 + \lambda)^{-1}V(\omega_0^2 + \lambda)^{-1}e^{-it\omega_0} f \right\|_{\mathfrak{h}}$$

$$\leq C \int_0^\infty d\lambda \lambda^{1/2}(m^2 + \lambda)^{-1} \left\| V(\omega_0^2 + \lambda)^{-1}e^{-it\omega_0} f \right\|_{\mathfrak{h}}$$

$$\leq C \|V\|_{L^2(\mathbb{R}^3)} \int_0^\infty d\lambda \lambda^{1/2}(m^2 + \lambda)^{-1} \left\| (\omega_0^2 + \lambda)^{-1}e^{-it\omega_0} f \right\|_{L^\infty(\mathbb{R}^3)}.$$

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Here, we have used the Hölder inequality in the last inequality. It follows from [6] that

$$\left\|(\omega_0^2 + \lambda)^{-1} e^{-it\omega_0} f\right\|_{L^\infty(\mathbb{R}^3)} \leq C|t|^{-3/2} \left\|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^1(\mathbb{R}^3)}$$

for any $t \in \mathbb{R}$. Since

$$
\begin{aligned}
&\left\|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^1(\mathbb{R}^3)} \\
&\leq \left\|(1+|x|)^{-2}(1+|x|)^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^1(\mathbb{R}^3)} \\
&\leq \left\|(1+|x|)^{-2}\right\|_{L^2(\mathbb{R}^3)} \left\|(1+|x|)^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^2(\mathbb{R}^3)} \\
&\leq C \left\|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^2(\mathbb{R}^3)} + C \left\||x|^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^2(\mathbb{R}^3)}
\end{aligned}
$$

and

$$|x|^2 (\omega_0^2 + \lambda)^{-1} = (\omega_0^2 + \lambda)^{-1}|x|^2 + 8(\omega_0^2 + \lambda)^{-3}\Delta - 6(\omega_0^2 + \lambda)^{-2},$$

we see that

$$
\begin{aligned}
&\left\|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\right\|_{L^1(\mathbb{R}^3)} \\
&\leq C(m^2 + \lambda)^{-1} \left\{ \left\|\omega_0^{5/2} f\right\|_{\mathfrak{h}} + \left\||x|^2 \omega_0^{5/2} f\right\|_{\mathfrak{h}} + \left\|\Delta \omega_0^{5/2} f\right\|_{\mathfrak{h}} \right\}.
\end{aligned}
$$

Thus, we obtain for any $t \in \mathbb{R}$,

$$\left\|(\omega - \omega_0) e^{-it\omega_0} f\right\|_{\mathfrak{h}} \leq C|t|^{-3/2} \int_0^\infty d\lambda \lambda^{1/2} (m^2 + \lambda)^{-2}$$

and hence (A.5) holds for any $f \in \mathcal{S}(\mathbb{R}^3)$.

A.3   Classical wave operator

We prove Lemma 2.6. It holds that

$$U(t)^* U_0(t) = \begin{bmatrix} U_{++}(t)^* e^{-it\omega_0} & U_{-+}(t)^* e^{it\omega_0} \\ U_{+-}(t)^* e^{-it\omega_0} & U_{--}(t)^* e^{it\omega_0} \end{bmatrix}.$$

For a, b, c, t and $s \in \mathbb{R}$, we set

$$I_{(a,b,c)}(s,t) = \omega_0^a e^{is\omega} \omega^b e^{it\omega_0} \omega_0^c.$$

If $a + b + c = 0$, then $I_{(a,b,c)}(s,t)$ is bounded and

$$\overline{I_{(a,b,c)}(s,t)} = I_{(a,b,c)}(-s,-t).$$

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By direct calculation, we have

$$
\begin{aligned}
U_{++}(t)^* e^{-it\omega_0} &= \overline{U_{--}(t)^* e^{it\omega_0}} \\
&= \frac{1}{4}\Big[I_{(1/2,0,-1/2)}(t,-t) + I_{(-1/2,0,1/2)}(t,-t) \\
&\quad + I_{(1/2,-1,1/2)}(t,-t) + I_{(-1/2,1,-1/2)}(t,-t) \\
&\quad + I_{(1/2,0,-1/2)}(-t,-t) + I_{(-1/2,0,1/2)}(-t,-t) \\
&\quad - I_{(1/2,-1,1/2)}(-t,-t) - I_{(-1/2,1,-1/2)}(-t,-t) \Big]
\end{aligned}
$$

and

$$
\begin{aligned}
U_{-+}(t)^* e^{it\omega_0} &= \overline{U_{+-}(t)^* e^{-it\omega_0}} \\
&= \frac{1}{4}\Big[I_{(-1/2,0,1/2)}(t,t) - I_{(1/2,0,-1/2)}(t,t) \\
&\quad + I_{(1/2,-1,1/2)}(t,t) - I_{(-1/2,1,-1/2)}(t,t) \\
&\quad + I_{(-1/2,0,1/2)}(-t,t) - I_{(1/2,0,-1/2)}(-t,t) \\
&\quad - I_{(1/2,-1,1/2)}(-t,t) + I_{(-1/2,1,-1/2)}(-t,t) \Big].
\end{aligned}
$$

By the similar argument as in the proof of Lemma 2.7 with the aid of Lemma 2.2, one can prove the following lemma:

Lemma A.2. *Suppose that Assumptions 1.1 and 1.2. Then:*

$$
\lim_{t\to\pm\infty} I_{(1/2,0,-1/2)}(t,-t) = \lim_{t\to\pm\infty} I_{(1/2,-1,1/2)}(t,-t) = \omega_0^{1/2} w_\pm \omega_0^{-1/2},
$$

$$
\lim_{t\to\pm\infty} I_{(-1/2,0,1/2)}(t,-t) = \lim_{t\to\pm\infty} I_{(-1/2,1,-1/2)}(t,-t) = \omega_0^{-1/2} w_\pm \omega_0^{1/2},
$$

$$
\lim_{t\to\pm\infty} I_{(1/2,0,-1/2)}(t,t) = \lim_{t\to\pm\infty} I_{(1/2,-1,1/2)}(t,t) = 0,
$$

$$
\lim_{t\to\pm\infty} I_{(-1/2,0,1/2)}(t,t) = \lim_{t\to\pm\infty} I_{(-1/2,1,-1/2)}(t,t) = 0.
$$

By the above lemma, we have

$$
\lim_{t\to\pm\infty} U_{++}(t)^* e^{-it\omega_0} = \frac{1}{2}\Big[\omega_0^{-1/2} w_\pm \omega_0^{+1/2} + \omega_0^{+1/2} w_\pm \omega_0^{-1/2} \Big],
$$

$$
\lim_{t\to\pm\infty} U_{-+}(t)^* e^{it\omega_0} = \frac{1}{2}\Big[\omega_0^{-1/2} w_\mp \omega_0^{+1/2} - \omega_0^{+1/2} w_\mp \omega_0^{-1/2} \Big].
$$

This completes the lemma.

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